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Renormalization Group Analysis of the Self-Avoiding Paths on the *d*-Dimensional Sierpiński Gaskets

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Notion of the renormalization group dynamical system, the self-avoiding fixed point and the critical trajectory are mathematically defined for the set of self-avoiding walks on the d-dimensional pre-Sierpiński gaskets (n-simplex lattices), such that their existence imply the asymptotic behaviors of the self-avoiding walks, such as the existence of the limit distributions of the scaled path lengths of "canonical ensemble," the connectivity constant (exponential growth of path numbers with respect to the length), and the exponent for mean square displacement. We apply the so defined framework to prove these asymptotic behaviors of the restricted self-avoiding walks on the 4-dimensional pre-Sierpiński gasket.

KEY WORDS: Renormalization group; self-avoiding walk; fractals; Sierpiński gasket.

1. INTRODUCTION AND MAIN RESULTS

1.1. Self-Avoiding Walk on the Sierpiński Gasket

Self-avoiding walks on hypercubic lattices $\mathbb{Z}^n$ have been studied mathematically for half a century, but compared to random walks (and diffusion processes, their continuum limits), amazingly little is known.[(1)]

For random walks, nice properties such as Markov properties enabled deep and accurate studies, many of which are effective for spaces with any dimension n. On the other hand, self-avoiding walks seem to have little such strong general methods. In fact, their behaviors are expected to vary drastically with the dimension n for small n, so that effective methods possibly vary for different spaces.

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Turning our attention to the 2- and 3-dimensional Sierpiński gaskets, there are works on the restricted self-avoiding walks (a subset of self-avoiding walks, to be defined in Section 3.1) in ref. 2, and mathematically rigorous studies for the full self-avoiding walk (including a proof that the restriced self-avoiding walk of ref. 2 are in the same universality classes with the full self-avoiding walk), with further precise asymptotic results, exist for both the 2-dimensional Sierpiński gasket[3–5] and the 3-dimensional Sierpiński gasket (4-simplex lattice).[6]

In the direction of generalization to d-dimensional Sierpiński gaskets, there is a work[7] on the restricted model for $d = 4, 5$, following the lines of ref. 2 with a propsal of an approximation method for general d. (d-dimensional Sierpiński gasket is the $d+1$-simplex lattice in ref. 7.) However, studies in the direction of extending the rigorous renormalization group analysis to d-dimensional cases have not appeared, to the authors' knowledge.

A main object of this paper to propose a general and mathematically rigorous renormalization group formulation of the self-avoiding walks on dSG for all d, from which one can derive asymptotic behaviors. As an application we prove asymptotic behaviors, such as the exponent for mean square displacement, of the restricted model of self-avoiding walks on 4SG. (The restricted model considers those self-avoiding walks which does not take 2 or more steps in row in each unit simplices (Section 3.1).)

We emphasize that a rigorous renormalization group analysis is non-trivial for the self-avoiding walks on dSG. Though it is easy to write down the renormalization group recursion equations for small d, it is of course another thing to analyze their trajectories rigorously. (Rigorous analysis of renormalization group trajectories and rigorous proofs of their implications on asymptotic behaviors of self-avoiding walks seem to have been ignored in the physics literature.)

It is not because the life is simple on gaskets that the gaskets are appealing, but because (as we will show in this paper) we can formulate and prove with mathematical rigor that an appropriate renormalization group formulation contains full imformation of asymptotic behaviors of self-avoiding walks. Since the renormalization group analysis contains full information on asymptotic behaviors, the authors think that it is too important not to analyze them with mathematical rigor and in generality (as we do in this paper).

1.2. Renormalization Group Approach

General "philosophy" of the renormalization group (RG) in physics (and the previous rigorous studies on 2SG and 3SG) suggest that a RG approach to the asymptotic behaviors of the self-avoiding paths on dSG starts with splitting the analysis into two parts:

(i) Formulate the RG, a dynamical system on a "natural" parameter space, and then derive nice properties about the fixed points and the trajectories of the RG flows, such as uniqueness of certain fixed point and convergence of critical trajectories.

(ii) Derive asymptotic behaviors of the self-avoiding paths from the properties of RG flows.

The RG is a dynamical system determined by a recursion map $\vec{\Phi}$, which will be defined in (10), on a finite dimensional Eulidean space (the parameter space $\mathbb{R}^{\mathscr{I}_d}$ defined in (3)). For general case of physical interest, we should consider infinite dimensional parameter space, but the so called finite ramifiedness of dSG implies that the RG in the present study is finite dimensional. The RG map is a response in the parameter space to the "scale transformation" (smoothing out or putting in finer structures to the paths) on the space of paths. (The transformation suitable for paths on dSG is a decimation, which will be implicit in the proof of Proposition 4.)

The quantities we need to extract from the RG map $\vec{\Phi}$ are the following.

(i) The largest eigenvalue λ of the differential map of $\vec{\Phi}$ at a self-avoiding fixed point $\vec{x}_c$.

(ii) The critical point β_c, which is the intersection point of the critical surface (the set of points from which the trajectories of RG converge to the self-avoiding fixed point $\vec{x}_c$) and the canonical curve (the curve defined by (17)).

We give the precise definitions of λ and β_c and also the assumptions on the RG map $\vec{\Phi}$ at (FP1)–(FP4) and (CS1) in Section 3.1. (To state them rigorously, we need to prepare technically cumbersome definitions in Section 2 starting from the definition of dSG.)

In this paper we will prove the following. Fix $d \geqq 2$. For each $k \in \mathbb{Z}_+$, let $N(k)$ be the number of k step self-avoiding paths on dSG starting from the origin O, and let $E_k[\,\cdot\,]$ be the expectation with respect to the uniform distribution (averaging with equal weight) on such paths.

Theorem 1 (Theorems 10 and 11). If there exists a critical point β_c then

(i) $\lim_{k\to\infty} \frac{1}{k} \log N(k) = \beta_c$.

(ii) $\lim_{k\to\infty} \frac{1}{\log k} \log E_k[|w(k)|^{s\,d_w}] = s$, $s \geqq 0$, where $|\cdot|$ denotes the Euclidean length and $d_w = \frac{\log\lambda}{\log 2}$.

The first result says that the connectivity constant of the self-avoiding paths on dSG is e^{β_c}. The second result says that the exponent for mean square displacement is $1/d_w$, which indicates that a typical k step self-avoiding path w deviates from the starting point by $|w(k)| \sim k^{1/d_w}$. (Since Theorem 11 holds for all $s \geqq 0$, we have the exponent for all the moments as well as that for the mean square displacement, but we will keep the good old terminology in this paper.) We will prove an additional statement on the correction to the "leading terms" $N(k) \sim e^{\beta_c k}$ and $|w(k)| \sim k^{1/d_w}$. See Theorems 10 and 11 for details.

Possibly the notions such as fixed points and critical points are not new from the view point of philosophy of RG. What is new here is that we propose a mathematically well-defined formulation (Section 3.1) which are sufficient (Section 3.2) to prove asymptotic behaviors of self-avoiding walks on dSG with all d, giving a mathematical evidence that the dynamics of RG contains information on the asymptotic behaviors of stochastic processes.

As an application of the formulation, we prove in Section 5 that the assumptions on the RG map in Section 3.1 are satisfied for the restricted model of self-avoiding paths on 4SG.

Theorem 2 (Theorems 14 and 15). The self-avoiding fixed point $\vec{x}_c$ and the critical point $\beta_{c,\,\mathrm{res}}$ of the restricted model on the 4 dimensional pre-Sierpiński gasket (4SG) exists.

In particular, the number $N_{\mathrm{res}}(k)$ of restricted self-avoiding paths of length k starting from 0 satisfies

$$\lim_{k\to\infty} \frac{1}{k} \log N_{\mathrm{res}}(k) = \beta_{c,\,\mathrm{res}}.$$

and the exponent for mean square displacement for the restricted model is the reciprocal of $d_w = \frac{\log\lambda}{\log 2} = 1.6657696\cdots$, in the sense that

$$\lim_{k\to\infty} \frac{1}{\log k} \log E_{\mathrm{res},\,k}[|w(k)|^{s\,d_w}] = s, \qquad s \geqq 0,$$

where $E_{\mathrm{res},\,k}$ is the expectation with respect to the probability measure with equal weight on length k restricted self-avoiding paths starting at O.

2. RENORMALIZATION GROUP

2.1. Self-Avoiding Paths on the *d*-Dimensional Pre-Sierpiński Gasket

Let $d \geqq 2$ be an integer. We define a d-dimensional pre-Sierpiński gasket (pre-dSG) as follows. Consider a d-simplex of a unit side length embedded in $\mathbb{R}^d$, and let $G_0 = \{v_0, v_1, v_2,..., v_d\}$ be the set of vertices of the d-simplex, where $v_0 = O = (0, 0,..., 0)$ is the origin of $\mathbb{R}^d$. (We may occasionally also write $v_{0,i} = v_i$, $i = 1,..., d$.) Let $B_0 = \{(v_i, v_j) \mid 0 \leqq i < j \leqq d\}$ be the set of non-ordered pairs of vertices, and we denote the pair (G_0, B_0) by F_0.

We define a sequence $F_n = (G_n, B_n)$, $n = 1, 2, 3,...$, of finite pre-dSG inductively by

$$G_{n+1} = \bigcup_{i=0}^{d} (G_n + 2^n v_i), \qquad B_{n+1} = \bigcup_{i=0}^{d} (B_n + 2^n v_i), \qquad n = 1, 2, 3,..., \tag{1}$$

where we write $A + v = \{x + v \mid x \in A\}$ for a set A and a point v.

F_n is a d-simplex of side length 2^n, composed of $d+1$ copies of F_{n-1}, with $d+1$ outmost points being $v_{n,0} = O$ and $v_{n,i} = 2^n v_i$, $i = 1, 2, 3,..., d$. G_n is a collection of vertices in the copies of G_{n-1}, and B_n is a collection of bonds in the copies of B_{n-1}.

We call

$$F = (G, B); \qquad G = \bigcup_{n=0}^{\infty} G_n, \qquad B = \bigcup_{n=0}^{\infty} B_n, \tag{2}$$

the d-dimensional pre-Sierpiński gasket (pre-dSG). We identify $(v, v') \in B$ with line segments $\overline{vv'}$ whenever it would be natural to do so.

Denote the set of non-negative integers by $\mathbb{Z}_+$, and for $w: \mathbb{Z}_+ \to G$, denote by $L(w) \in \mathbb{Z}_+ \cup \{\infty\}$ ("the length of w") the smallest integer satisfying

$$w(i) = w(L(w)), \qquad i \geqq L(w).$$

Define the set of self-avoiding paths W_0 to be the set of maps $w: \mathbb{Z}_+ \to G$, such that

$$\begin{aligned} &w(i_1) \neq w(i_2), && 0 \leqq i_1 < i_2 \leqq L(w), \\ &|w(i) - w(i+1)| = 1, && 0 \leqq i \leqq L(w) - 1, \\ &\overline{w(i)\, w(i+1)} \in B, && 0 \leqq i \leqq L(w) - 1. \end{aligned}$$

2.2. Overview of Technical Definitions

We need to prepare several basic definitions in Sections 2.3, 2.4, and 2.5 before introducing the main notions in Section 3.1. Here we will briefly explain the basic definitions.

Section 2.3. We first classify how a self-avoiding path intersects a unit d-simplex. A path which enters a simplex moves within the simplex for at most d steps (because it may not hit the same vertex twice). If a path takes i_1 steps in the simplex and goes into an adjacent one, and never returns to the simplex, we label the intersection of the path and the simplex by the index $(i_1, 0,..., 0)$. Alternatively, the path may return to the simplex a number of times, and for each return the intersection may be labelled by how many steps the path takes in the simplex. Thus if a path spends 3 steps for the first intersection and 1 step for the second intersection with a simplex, then we label the intersection by the index $(1, 3, 0,..., 0)$. (For our purpose we may identify $(1, 3, 0,..., 0)$ and $(3, 1, 0,..., 0)$; we are free to rearrange a sequence in an index in the ascending order.) We denote the set of the indices by $\mathscr{I}_d$.

Each index corresponds to a component in the parameter space on which the RG map acts. Therefore for each index $I \in \mathscr{I}_d$, we need a set of self-avoiding paths $W_I^{(n)}$ on G_n labelled by I which has a similar structure as the intersection of a path and a unit simplex labelled by I. For an index with more than one non-zero entries, such as $I = (1, 3, 0,..., 0)$, the set $W_I^{(n)}$ is defined to be a set of collection of self- and mutually-avoiding paths on G_n. For example, $W^{(n)}_{(1, 3, 0,..., 0)}$ is a set of disjoint pairs of self-avoiding paths on G_n, such that one path starts and ends at outmost vertices of G_n, but hits no other outmost vertices, while the other path hits two outmost vertices other than the endpoints.

Section 2.4. The RG in our study is the recursion map in n of the joint generating functions $\vec{X}_n = (X_{n,I}(\vec{x}), I \in \mathscr{I}_d)$ of s_J, $J \in \mathscr{I}_d$, for $W_I^{(n)}$, where s_J is the number of unit simplices whose intersection with the path is of type J.

A similarity of finite gaskets G_n among different ns implies a recursion relation to hold for all n, and this is our RG. In this way we arrive at a mathematically well-defined notion of "a response in the parameter space of the scale transformation in the path space."

Section 2.5. A study in 3SG shows[6] that in general there are more than one non-trivial fixed points of the RG. Therefore we have to know which fixed point is relevant for the asymptotic behavior of the self-avoiding paths. It turns out that the condition that the fixed point is in a certain invariant set of the RG ensures our proof to work. To formulate the condition (see (FP4)), we introduce the invariant set Ξ_d.

We note that it would also be useful for intuitive understanding to look at the case of 3SG, which is explicitly given in ref. 6.

2.3. Classification of Self-Avoiding Paths

Denote by $\mathcal{T}_b$ the family of all the translations of B_0 that are subsets of B. Namely, $\mathcal{T}_b$ contains all the unit d-simplices which compose the pre-dSG. (with each simplex regarded as a collection of bonds). Put

$$\begin{aligned}\mathcal{I}_d = \{(i_1, i_2,..., i_k) \in \mathbb{Z}_+^k \mid k = 1, 2, 3,..., 0 < i_1 \leqq i_2 \leqq \cdots \leqq i_k,\\ i_1 + i_2 + \cdots + i_k + k \leqq d + 1\},\end{aligned} \tag{3}$$

and denote the number of elements of $\mathcal{I}_d$ by $f_d = \sharp \mathcal{I}_d$.

Proposition 3. Let $w \in W_0$ and $\Delta \in \mathcal{T}_b$, and consider the set of bonds

$$A = \{\overline{w(i)\, w(i+1)} \in \Delta \mid i = 0, 1, 2,..., L(w)\}.$$

If A is not empty, then there exists $I = (i_1, i_2,..., i_k) \in \mathcal{I}_d$ such that A is congruent to

$$\Delta_I = \{\overline{Ov_1 v_2 \cdots v_{i_1 - 1} v_{i_1}}, \overline{v_{i_1+1} \cdots v_{i_1+i_2}},..., \overline{v_{i_1+\cdots i_{k-1}+1} \cdots v_{i_1+\cdots i_k}}\}, \tag{4}$$

where we used an abbreviation such as

$$\overline{Ov_1 v_2 \cdots v_{i_1-1} v_{i_1}} = \overline{Ov_1}, \overline{v_1 v_2}, \overline{v_2 v_3},..., \overline{v_{i_1-1} v_{i_1}}.$$

Example.

- $\mathcal{I}_2 = \{(1), (2)\}$: $A \neq \emptyset$ is congruent to either $\{\overline{Ov_1}\}$ or $\{\overline{Ov_1 v_2}\}$.
- $\mathcal{I}_3 = \{(1), (2), (3), (1, 1)\}$: There is a possibility that a path enters a unit tetrahedron twice, as $\{\overline{Ov_1}, \overline{v_2 v_3}\}$.
- $\mathcal{I}_4 = \{(1), (2), (3), (4), (1, 1), (1, 2)\}$.

Correspondingly, $f_2 = 2$, $f_3 = 4$, $f_4 = 6$.

Proof of Proposition 3. If $A \neq \emptyset$, namely, if the path w enters the unit d-simplex specified by Δ, then A is composed of one or more connected clusters. That is, w may pass through Δ and may come back and reenter Δ. Since w is self-avoiding, the second passage does not intersect with the first one. Thus we can classify A by the size of the connected segments. One may rearrange the segments in an increasing order of size, hence each class is determined by an increasing finite sequence of positive integers, $i_1 \leqq i_2 \leqq i_3 \leqq \cdots \leqq i_k$ for some $k \geqq 1$. The meaning of the conditions in the definition of $\mathscr{I}_d$ should now be obvious. Since Δ is a translation of B_0 which is the set of bonds in the unit d-simplex $Ov_1v_2 \cdots v_d$, the statement follows. ▮

In analogy with Proposition 3 we can classify the set of self-avoiding paths on F_n by $\mathscr{I}_d$, and also generalize to two or more self-avoiding paths.

For $n \in \mathbb{Z}_+$ and $u, v \in G_n$, define $W^{(n,u,v)}$ by

$$W^{(n,u,v)} = \{w \in W_0 \mid w(0) = u, w(L(w)) = v, w(i) \in G_n, i \in \mathbb{Z}_+\}.$$

For $n \in \mathbb{Z}_+$ and $I = (i_1, i_2, \ldots, i_k) \in \mathscr{I}_d$, define $W_I^{(n)}$ by

$$\begin{aligned} W_I^{(n)} = \{ & (w_1, w_2, \ldots, w_k) \in W^{(n, O, v_{n,i_1})} \times W^{(n, v_{n,i_1+1}, v_{n,i_1+i_2+1})} \\ & \times W^{(n, v_{n,i_1+i_2+2}, v_{n,i_1+i_2+i_3+2})} \times \cdots \times W^{(n, v_{n,i_1+i_2+\cdots+i_{k-1}+k-1}, v_{n,i_1+i_2+\cdots+i_k+k-1})} \mid \\ & \text{if } i \neq j \text{ then } w_i \text{ and } w_j \text{ do not hit common points, and for each } j \\ & w_j \text{ hits points } v_{n,i_1+i_2+\cdots+i_{j-1}+j-1}, v_{n,i_1+i_2+\cdots+i_{j-1}+j}, v_{n,i_1+i_2+\cdots+i_{j-1}+j+1}, \\ & \cdots v_{n,i_1+i_2+\cdots+i_{j-1}+i_j+j-1}, \text{ in this order,} \\ & \text{but hits no other points in } \{v_{n,\ell} \mid \ell = 0, 1, 2, \ldots, d\}\}. \end{aligned} \tag{5}$$

Obviously, k is equal to the number of path segments that form an element in $W_I^{(n)}$.

Example. For $d = 4$, there are $f_4 = 6$ types of sets $W_I^{(n)}$, which are

$\{(1)\}$: Set of paths from O to $v_{n,1}$ which do not hit $v_{n,2}, v_{n,3}, v_{n,4}$.

$\{(2)\}$: Set of paths from O to $v_{n,2}$ passing through $v_{n,1}$ which do not hit $v_{n,3}, v_{n,4}$.

$\{(3)\}$: Set of paths from O to $v_{n,3}$ passing through $v_{n,1}$ and $v_{n,2}$ in this order and avoiding $v_{n,4}$.

$\{(4)\}$: Set of paths from O to $v_{n,4}$ passing through $v_{n,1}, v_{n,2}$, and $v_{n,3}$ in this order.

$\{(1, 1)\}$: Set of pair of (self- and mutually-avoiding) paths, one from O to $v_{n,1}$ and the other from $v_{n,2}$ to $v_{n,3}$ neither hitting $v_{n,4}$.

$\{(1, 2)\}$: Set of pair of paths, one from O to $v_{n,1}$ and the other from $v_{n,2}$ to $v_{n,4}$ via $v_{n,3}$.

For $w \in \bigcup_{n \in \mathbb{Z}_+} \bigcup_{I \in \mathcal{I}_d} W_I^{(n)}$ denote by $\hat{w}$ the set of bonds which w passes. Namely, for $n \in \mathbb{Z}_+$ and $I = (i_1, i_2, \ldots, i_k) \in \mathcal{I}_d$, and for $w = (w_1, w_2, \ldots, w_k) \in W_I^{(n)}$,

$$\hat{w} = \{\overline{w_j(i)\, w_j(i+1)} \in B \mid i = 0, 1, 2, \ldots, L(w_j) - 1,\ j = 1, 2, \ldots, k\}.$$

Also define $S_I(w)$, $I \in \mathcal{I}_d$, by,

$$S_I(w) = \{\Delta \in \mathcal{T}_b \mid \hat{w} \cap \Delta \text{ is congruent to } \Delta_I \text{ of (4)}\}, \tag{6}$$

and denote by $s_I(w) = \sharp S_I(w)$, the cardinality of $S_I(w)$. $s_I(w)$ is the number of unit d-simplices in F such that the trajectory of the path (or the paths) w is congruent to Δ_I. It is a generalized notion of the length of the path in the sense that

$$\sum_{i=1}^{k} L(w_i) = \sum_{J \in \mathcal{I}_d} |J|\, s_J(w), \qquad w = (w_1, w_2, \ldots, w_k) \in W_I^{(n)},$$
$$I = (i_1, \ldots, i_k) \in \mathcal{I}_d,\ n \in \mathbb{Z}_+, \tag{7}$$

where, for $J \in \mathcal{I}_d$ we define $|J|$, the length of J, by

$$|J| = j_1 + \cdots + j_\ell, \qquad \text{if} \quad J = (j_1, \ldots, j_\ell). \tag{8}$$

2.4. Parameter Space and the Renormalization Group

Assumptions of the main results are stated in terms of the flows of the associated renormalization group (RG), which is a map (discrete-time dynamical system) in a parameter space of variables in the generating function of generalized path length (s_J, $J \in \mathcal{I}_d$). The dynamical system is derived as the response in the parameter space to the change in n. A graphical property of dSG called finite ramifideness implies that the RG is a finite dimensional dynamical system.

Define the generating function

$$\vec{X}_n = (X_{n,I}, I \in \mathcal{I}_d) \colon \mathbb{C}^{\mathcal{I}_d} \to \mathbb{C}^{\mathcal{I}_d}$$

of (s_J, $J \in \mathcal{I}_d$) for a family of paths sets ($W_I^{(n)}$, $I \in \mathcal{I}_d$), by,

$$X_{n,I}(\vec{x}) = \sum_{w \in W_I^{(n)}} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w)}, \qquad \vec{x} = (x_J, J \in \mathcal{I}_d) \in \mathbb{C}^{\mathcal{I}_d}, \qquad n = 0, 1, 2, \ldots. \tag{9}$$

The right hand side is a finite summation, so $X_{n,I}$ is defined on $\mathbb{C}^{\mathcal{I}_d}$.

The starting point of our analysis is the following.

Proposition 4. $\vec{X}_n = (X_{n,I}, I \in \mathcal{I}_d)$, $n = 0, 1, 2, \ldots$, satisfy the following recursion relations.

$$\vec{X}_0(\vec{x}) = \vec{x},\ \vec{x} \in \mathbb{C}^{\mathcal{I}_d},$$

and

$$\vec{X}_{n+1} = \vec{\Phi} \circ \vec{X}_n, \tag{10}$$

where

$$\vec{\Phi} = (\Phi_I, I \in \mathcal{I}_d) = \vec{X}_1,$$

is a f_d dimensional vector valued function whose components are polynomials in f_d variables with positive integer coefficients. In particular, $\mathbb{R}_+^{\mathcal{I}_d}$ is an invariant set of $\vec{\Phi}$.

The degree of each term in the polynomials are no less than 2 and no greater than $d+1$, and $\Phi_{(1)}$ contains terms ${x_{(1)}}^2$ and ${x_{(1)}}^{d+1}$.

Proof. Let $n \in \mathbb{Z}_+$. F_1 is composed of $d+1$ d-simplices congruent to F_0. Similarly, F_{n+1} is composed of $d+1$ d-simplices F_n. The similarity of the two compositions leads to a natural map

$$\pi\colon W_I^{(n+1)} \to W_I^{(1)}, \qquad I \in \mathcal{I}_d.$$

For each $X_{n+1,I}$, classify the summation in the right hand side of (9) (with $n+1$ in place of n) by $\pi(w) \in W_I^{(1)}$ to find

$$\begin{aligned} X_{n+1,I}(\vec{x}) &= \sum_{w \in W_I^{(n+1)}} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w)} = \sum_{w' \in W_I^{(1)}} \sum_{w \in W_I^{(n+1)};\, \pi(w) = w'} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w)} \\ &= \sum_{w' \in W_I^{(1)}} \prod_{I' \in \mathcal{I}_d} \Bigg(\sum_{w'' \in W_{I'}^{(n)}} \prod_{J \in \mathcal{I}_d} x_J^{s_J(w'')} \Bigg)^{s_{I'}(w')} \\ &= \sum_{w' \in W_I^{(1)}} \prod_{I' \in \mathcal{I}_d} (X_{n,I'}(\vec{x}))^{s_{I'}(w')} = X_{1,I}(\vec{X}_n(\vec{x})). \end{aligned}$$

By definition (9), each term in $\Phi_I = X_{1,I}$ has a form $\prod_{J \in \mathcal{I}_d} x_J^{s_J(w)}$, hence its degree $\sum_{J \in \mathcal{I}_d} s_J(w)$ is, by definition (6), the number of unit simplices in F_1 that a path w passes through. This is bounded from above by the total number of unit simplices in F_1, which is $d+1$, and from below by 2, because any two extreme (outmost) vertices of F_1 is apart by length 2.

Positivity of coefficients of $X_{1,I}$ are obvious. Existence of terms $x_{(1)}^2$ and $x_{(1)}^{d+1}$ in $\Phi_{(1)} = X_{1,(1)}$ follows from the paths $\overline{Ov_{0,1}v_{1,1}}$ and $\overline{Ov_{0,d}(v_{0,d}+v_{0,d-1})(v_{0,d-1}+v_{0,d-2})\cdots(v_{0,2}+v_{0,1})\,v_{1,1}}$ in $W_{(1)}^{(1)}$. ∎

Large n means that the endpoints of the paths are far apart, hence it corresponds to large path length L. Intuitively speaking Proposition 4 therefore gives a response to the change in the length scale of the system in consideration, the sets of self-avoiding paths, in terms of the parameter space of variables in the generating functions of s_I, the generalized path length. Global properties of the trajectories of the map $\vec{\Phi}$ therefore is expected to give (and we will show that it does) large length asymptotic behaviors of self-avoiding paths on dSG.

In analogy to the (mathematically misleading) terminology in physics literature, we call the discrete-time dynamical system on $\mathbb{R}_+^{\mathscr{I}_d}$ defined by the map $\vec{\Phi}$, the renormalization group (RG).

2.5. Invariant Sets

If there is a subset of $\mathbb{R}_+^{\mathscr{I}_d}$ which is an invariant set of the RG map $\vec{\Phi}$, then the recursion (10) is naturally regarded as a recursion equation on the subset.

For $I = (i_1,\ldots, i_k)$ and $J = (j_1,\ldots, j_\ell)$ in $\mathscr{I}_d$, denote by $I \oplus J$ the rearrangement of $i_1,\ldots, i_k, j_1,\ldots, j_\ell$ in non-decreasing order, and define $\Xi_d \subset \mathbb{R}_+^{\mathscr{I}_d}$ by

$$\Xi_d = \{\vec{x} \in \mathbb{R}_+^{\mathscr{I}_d} \mid x_{I\oplus J} \leqq x_I x_J \text{ for all } I, J \in \mathscr{I}_d \text{ such that } I \oplus J \in \mathscr{I}_d\}. \quad (11)$$

Example. $\Xi_3 = \{\vec{x} \in \mathbb{R}_+^{\mathscr{I}_3} \mid x_{(11)} \leqq x_{(1)}^2\}$, $\Xi_4 = \{\vec{x} \in \mathbb{R}_+^{\mathscr{I}_4} \mid x_{(11)} \leqq x_{(1)}^2$, $x_{(12)} \leqq x_{(1)} x_{(2)}\}$.

Proposition 5. Ξ_d is an invariant set of $\vec{\Phi}$.

Proof. Let $I, J, I\oplus J \in \mathscr{I}_d$. Note that there is a natural one-to-one into map $W_{I\oplus J}^{(1)} \to W_I^{(1)} \times W_J^{(1)}$. For $w \in W_{I\oplus J}^{(1)}$, let $(w_1, w_2) \in W_I^{(1)} \times W_J^{(1)}$ be the corresponding pair. Then, for each $\Delta \in \mathscr{T}_b$, $\hat{w} \cap \Delta$ may be regarded as a composition of $\hat{w}_1 \cap \Delta$ and $\hat{w}_2 \cap \Delta$, hence if $\hat{w} \cap \Delta$ is congruent to Δ_K of (4) for some $K \in \mathscr{I}_d$, then there exists $K_1, K_2 \in \mathscr{I}_d$ (allowing an emptyset) such that $K = K_1 \oplus K_2$ and such that $\hat{w}_i \cap \Delta$, $i = 1, 2$, is congruent to Δ_{K_i}, $i = 1, 2$, respectively. Note also that $\vec{x} \in \Xi_d$ implies $x_{K_1\oplus K_2} \leqq x_{K_1} x_{K_2}$.

Therefore by definition (Proposition 4 and (9)),

$$\begin{aligned}\Phi_{I\oplus J}(\vec{x}) &= \sum_{w\in W^{(1)}_{I\oplus J}} \prod_{K\in\mathscr{I}_d} x_K{}^{s_K(w)}\\ &\leqq \sum_{w_1\in W^{(1)}_I} \sum_{w_2\in W^{(1)}_J} \prod_{K_1\in\mathscr{I}_d} x_{K_1}{}^{s_{K_1}(w_1)} \prod_{K_2\in\mathscr{I}_d} x_{K_2}{}^{s_{K_2}(w_2)} = \Phi_I(\vec{x})\,\Phi_J(\vec{x}). \quad \blacksquare\end{aligned}$$

In the following, for $\mathscr{K}\subset\mathscr{I}_d$, we use a (somewhat irregular) notation

$$\mathbb{R}_+^{\mathscr{K}} = \{\vec{x}\in\mathbb{R}_+^{\mathscr{I}_d} \mid x_J = 0,\, J\notin\mathscr{K}\} \subset \mathbb{R}_+^{\mathscr{I}_d}. \tag{12}$$

We also write $\mathbb{C}^{\mathscr{K}}\subset\mathbb{C}^{\mathscr{I}_d}$, $\mathbb{Z}_+^{\mathscr{K}}\subset\mathbb{Z}_+^{\mathscr{I}_d}$, etc.

Define

$$\mathscr{K}_{\mathrm{res}} = \{(1), (11), \ldots, (1\cdots 1)\}. \tag{13}$$

The indices in $\mathscr{K}_{\mathrm{res}}$ correspond to those paths which go out of a simplex after single step passage each time they enter the simplex.

Proposition 6. $\mathbb{R}_+^{\mathscr{K}_{\mathrm{res}}}$ is an invariant subset of $\vec{\Phi}$.

Proof. This is proved by generalizing the arguments in the proof of ref. 6, Proposition 2.1 (4)(5). ∎

3. MAIN RESULTS

3.1. Fixed Point and Critical Trajectory

Based on experiences with dSG for $d = 2, 3, 4$, we define notions which are relevant for asymptotic behaviors of self-avoiding paths on dSG.

Denote the Jacobi matrix of $\vec{\Phi}$ in Proposition 4 by $\mathscr{J} = (\mathscr{J}_{IJ})$:

$$\mathscr{J}_{IJ}(\vec{x}) = \frac{\partial\Phi_I}{\partial x_J}(\vec{x}), \qquad I, J\in\mathscr{I}_d, \quad \vec{x}\in\mathbb{C}^{\mathscr{I}_d}. \tag{14}$$

We say that $\vec{x}_c\in\mathbb{R}_+^{\mathscr{I}_d}$ is a *self-avoiding fixed point*, if the following hold.

(FP1) $\vec{\Phi}(\vec{x}_c) = \vec{x}_c$.

(FP2) $\mathscr{J}(\vec{x}_c)$ in (14) is diagonalizable by an invertible matrix. The eigenvalue λ which is largest in absolute value satisfies $\lambda > 1$ with multiplicity 1, and all the other eigenvalues have absolute values strictly less than 1.

Denote by $\vec{v}_L = (v_{L,I}, I\in\mathscr{I}_d)$ a left eigenvector of $\mathscr{J}(\vec{x}_c)$ corresponding to λ;

$$\sum_{I\in\mathscr{I}_d} v_{L,I}\mathscr{J}_{IJ}(\vec{x}_c) = \lambda v_{L,J}, \qquad J\in\mathscr{I}_d,$$

which we chose to have non-negative components (possible, thanks to Frobenius' theorem). Then $v_{L,J} > 0$, $J \in \mathscr{I}_d$.

Similarly, denote by $\vec{v}_R$ a right eigenvector corresponding to λ with non-negative components;

$$\sum_{J \in \mathscr{I}_d} \mathscr{J}_{IJ}(\vec{x}_c)\, v_{R,J} = \lambda v_{R,I}, \qquad I \in \mathscr{I}_d.$$

Then $v_{R,(1)} > 0$.

(FP3) For all $I \in \mathscr{I}_d$ such that $x_{c,I} \neq 0$, there exists $m \in \mathbb{Z}_+^{\mathscr{I}_d}$, satisfying $m_{(1)} > 0$ and $m_J = 0$ if $x_{c,J} = 0$, such that there is a term $\prod_{J \in \mathscr{I}_d} x_J{}^{m_J}$ in Φ_I.

(FP4) $\vec{x}_c \in \Xi_d \setminus \{\vec{0}\}$.

Assume that there exists a self-avoiding fixed point $\vec{x}_c$. We say that $\vec{x} \in \mathbb{R}_+^{\mathscr{I}_d}$ is *in the domain of attraction of* $\vec{x}_c$, if the following hold.

(DA1) $\lim_{n \to \infty} \vec{X}_n(\vec{x}) = \vec{x}_c$.

(DA2) If $x_{c,I} \neq 0$ then $x_I \neq 0$.

We denote by $\mathscr{D}om(\vec{x}_c)$ the set of $\vec{x} \in \mathbb{R}_+^{\mathscr{I}_d}$ which are in the domain of attraction of $\vec{x}_c$.

Example. If $\vec{x}_c$ is a self-avoiding fixed point, then $\vec{x}_c \in \mathscr{D}om(\vec{x}_c)$; i.e., a self-avoiding fixed point satisfies (DA1)–(DA2).

Let $\mathscr{K} \subset \mathscr{I}_d$. Instead of (5), we may consider a set of walks $W^{(n)}_{\mathscr{K},I}$ by restricting to those paths in $W^{(n)}_I$ which satisfy $s_J(w) = 0$ if $J \notin \mathscr{K}$:

$$W^{(n)}_{\mathscr{K},I} = \{ w \in W^{(n)}_I \mid s_J(w) = 0,\ J \notin \mathscr{K} \}. \tag{15}$$

If $\mathscr{K} = \mathscr{I}_d$, then we are dealing with the original (full) model; $W^{(n)}_{\mathscr{I}_d,I} = W^{(n)}_I$. We define the corresponding generating functions by

$$X_{\mathscr{K},n,I}(\vec{x}) = \sum_{w \in W^{(n)}_{\mathscr{K},I}} \prod_{J \in \mathscr{K}} x_J^{s_J(w)},$$
$$\vec{x} = (x_J,\ J \in \mathscr{K}) \in \mathbb{C}^{\mathscr{K}}, \quad n = 0, 1, 2,..., \quad I \in \mathscr{I}_d. \tag{16}$$

If $\mathbb{R}_+^{\mathscr{K}}$ is an invariant subset of $\mathbb{R}_+^{\mathscr{I}_d}$, then $\vec{X}_{\mathscr{K},n}$ satisfy (10), with a convention that the components corresponding to $J \notin \mathscr{K}$ are 0.

For $\mathscr{K} \subset \mathscr{I}_d$ and $\beta \in \mathbb{R}$, define $\vec{x}_{\mathrm{can},\mathscr{K}}(\beta) = (x_{\mathrm{can},\mathscr{K},I}(\beta), I \in \mathscr{I}_d)$ by

$$x_{\mathrm{can},\mathscr{K},I}(\beta) = \begin{cases} e^{-\beta |I|}, & I \in \mathscr{K}, \\ 0, & I \notin \mathscr{K}, \end{cases} \tag{17}$$

where $|I|$ is defined in (8). Following the notions in statistical mechanics, the partition function for a set of self-avoiding paths specified by $\mathscr{K}$ is defined by $\vec{Z}_{\mathscr{K},n} = (Z_{\mathscr{K},n,I}, I \in \mathscr{I}_d)$, with

$$Z_{\mathscr{K},n,I}(\beta) = \sum_{w \in W^{(n)}_{\mathscr{K},I}} e^{-\beta L(w)}, \qquad \beta \in \mathbb{R}, \quad n = 0, 1, 2, \ldots. \tag{18}$$

With (7), we see that

$$Z_{\mathscr{K},n,I}(\beta) = X_{\mathscr{K},n,I}(\vec{x}_{\mathrm{can},\mathscr{K}}(\beta)). \tag{19}$$

In view of this relation, we will occasionally refer to the curve in the parameter space $\mathbb{R}^{\mathscr{I}_d}$ defined by (17) as the "canonical curve."

If $\mathscr{K} = \mathscr{I}_d$ we also use an abbreviation $\vec{x}_{\mathrm{can}}(\beta) = \vec{x}_{\mathrm{can},\mathscr{I}_d}(\beta)$ and $\vec{Z}_n(\beta) = \vec{Z}_{\mathscr{I}_d,n}(\beta)$. Hence

$$Z_{n,I}(\beta) = X_{n,I}(\vec{x}_{\mathrm{can}}(\beta)) = \sum_{w \in W^{(n)}_{I}} e^{-\beta L(w)}, \qquad \beta \in \mathbb{R}, \quad n = 0, 1, 2, \ldots. \tag{20}$$

In the following, the set of paths in (15) with $\mathscr{K} = \mathscr{K}_{\mathrm{res}}$ will be called the restricted self-avoiding paths, the corresponding generating function (16), the generating function for the restricted model, and so on.

We need the following additional definitions for Theorem 10.

(CS1) We say that $\beta_c \in \mathbb{R}$ is a critical point of the full model if $\vec{x}_{\mathrm{can}}(\beta_c) \in \mathscr{D}om(\vec{x}_c)$ for a self-avoiding fixed point $\vec{x}_c$.

(CS2) We say that $\beta_{c,\mathrm{res}} \in \mathbb{R}$ is a critical point of the restricted model if $\vec{x}_{\mathrm{can},\mathscr{K}_{\mathrm{res}}}(\beta_{c,\mathrm{res}}) \in \mathscr{D}om(\vec{x}_c)$ for a self-avoiding fixed point $\vec{x}_c$.

Note that by definition (11),

$$\vec{x}_{\mathrm{can}}(\beta_c) \in \Xi_d, \qquad \text{and} \qquad \vec{x}_{\mathrm{can},\mathscr{K}_{\mathrm{res}}}(\beta_{c,\mathrm{res}}) \in \Xi_d. \tag{21}$$

Remarks. (i) The boundary ∂D of the set $D \subset \Xi_d$ defined in (27) is a bounded closed non-empty $\vec{\Phi}$-invariant subset of $\mathbb{R}^{\mathscr{I}_d}$ (which are easy consequences of Theorem 12). Hence, the fixed point theorem implies that there exists a fixed point of $\vec{\Phi}$ which satisfies (FP1) and (FP4).

The other conditions on the self-avoiding fixed point (FP2) and (FP3) depend more on the details of the self-avoiding paths on dSG. However, these conditions deal with conditions of Perron–Frobenius type and irreducibility, which are "soft" conditions, hence we expect them to hold.

(ii) What may be more difficult is the condition (CS1), which states existence of a trajectory converging to a fixed point. This essentially suggests that a bounded trajectory necessary converges to a fixed point (at least in the domain Ξ_d), that the renormalization group dynamical system is free of limit cycles much less any chaotic behaviors. There are of course many discrete dynamical systems, even on one-dimensional space, which exhibit chaotic behaviors, hence this condition is far from trivial.

On the other hand, it is proved in refs. 3 and 6 that for $d = 2$ and $d = 3$, all the conditions (FP1)–(FP4) and (CS1) are satisfied. We also prove in Section 5 that (FP1)–(FP4) and (CS2) are satisfied for the restricted model on 4SG. Based on these results, we conjecture that these conditions are satisfied (hence the results about the asymptotic behaviors of the self-avoiding walks hold) for all d.

3.2. Asymptotic Behaviors

Here we will state main consequences of assumptions on RG formulated in Section 3.1.

First we note the following characterization of a critical point β_c.

Theorem 7. If $\beta_c \in \mathbb{R}$ is a critical point of the full model and $\vec{x}_c$ the corresponding fixed point (implicit in the definition (CS1)), then for $I \in \mathscr{I}_d$,

$$\lim_{n\to\infty} Z_{n,I}(\beta) = \begin{cases} 0, & \beta > \beta_c, \\ x_{c,I}, & \beta = \beta_c. \end{cases}$$

Moreover,

$$\lim_{n\to\infty} \sum_{i=1}^{d} Z_{n,(i)}(\beta) = \infty, \qquad \beta < \beta_c.$$

In particular, critical point (if exists) is unique.

Similar result holds also for the restricted model (CS2).

Since the critical point (if exists) is unique, there is one and only one self-avoiding fixed point that is related by (CS1) to the critical point.

Though it is not trivial to prove the uniqueness of self-avoiding fixed point, we therefore can (and we will, in the proof of Theorem 10) talk about the unique self-avoiding fixed point that is related to the critical point, under the asumption that the critical point exists.

To state the next Theorem, we note a relation between 0 components of a fixed point and an invariant subset of $\vec{\Phi}$. In the known case of $d = 2$ and $d = 3$, the self-avoiding fixed points have 0 components. We will write

$$\mathscr{K}_{\vec{x}_c} = \{I \in \mathscr{I}_d \mid x_{c,I} \neq 0\}, \tag{22}$$

and, as in (12),

$$\mathbb{R}_+^{\mathscr{K}_{\vec{x}_c}} = \{\vec{x} \in \mathbb{R}_+^{\mathscr{I}_d} \mid x_J = 0, J \notin \mathscr{K}_{\vec{x}_c}\} .$$

Proposition 8. $\mathbb{R}_+^{\mathscr{K}_{\vec{x}_c}}$ is an invariant subset of $\vec{\Phi}$.

Proof. If $x_{c,I} = 0$ then $\Phi_I(\vec{x}_c) = x_{c,I} = 0$. On the other hand Φ_I is a polynomial with positive coefficients. Therefore, each term in $\Phi_I(\vec{x})$ contains one of x_J's such that $x_{c,J} = 0$. In other words, each term in $\Phi_I(\vec{x})$ contains x_J such that $J \notin \mathscr{K}_{\vec{x}_c}$.

Therefore, if $\vec{x} \in \mathbb{R}_+^{\mathscr{K}_{\vec{x}_c}}$, then $\Phi_I(\vec{x}) = 0$ for those I satisfying $x_{c,I} = 0$, or equivalently, $I \notin \mathscr{K}_{\vec{x}_c}$. ▮

Remark. For $d = 2$ and $d = 3$, the results in refs. 3 and 6 respectively proves (by explicit calculations) that the self-avoiding fixed point $\vec{x}_c$ is unique and that $\mathscr{K}_{\text{res}} = \mathscr{K}_{\vec{x}_c}$.

For $I \in \mathscr{I}_d$, $n \in \mathbb{Z}_+$, and $\vec{x} \in \mathbb{R}_+^{\mathscr{I}_d}$, define a probability measure $\mu_{\vec{x}, n, I}$ on the finite set $W_I^{(n)}$ by

$$\mu_{\vec{x}, n, I}[\{w\}] = \frac{1}{X_{n,I}(\vec{x})} \prod_{J \in \mathscr{I}_d} x_J^{s_J(w)}, \qquad w \in W_I^{(n)}, \tag{23}$$

whenever $X_{n,I}(\vec{x}) \neq 0$.

Note that if $x_{c,I} \neq 0$ and $\vec{x} \in \mathscr{D}om(\vec{x}_c)$, then (DA1) implies that $X_{n,I}(\vec{x}) > 0$ for sufficiently large n, hence if $\vec{x} \in \mathscr{D}om(\vec{x}_c)$ then $\mu_{\vec{x}, n, I}$ is well defined.

Theorem 9. Let $\vec{x}_c$ be a self-avoiding fixed point and $\vec{x} \in \mathscr{D}om(\vec{x}_c)$. Then the following hold.

(i) There exists $f_d \times f_d$ matrix $\Lambda(\vec{x})$ whose elements are non-negative such that

$$\Lambda(\vec{x}) = \lim_{n \to \infty} \lambda^{-n} \mathscr{I}_n(\vec{x}), \tag{24}$$

where $\mathscr{I}_n$ is as in (14).

(ii) For $I \in \mathscr{K}_{\vec{x}_c}$, the joint distribution of scaled generalized path lengths $(\lambda^{-n} s_J, J \in \mathscr{K}_{\vec{x}_c})$ under $\mu_{\vec{x}, n, I}$ converges weakly to a Borel probability measure $p^*_{\vec{x}, I}$ on $\mathbb{R}^{\mathscr{I}_d}$ as $n \to \infty$. Here, λ is as in (FP2). $p^*_{\vec{x}, I}$ is supported on $\mathbb{R}_+^{\mathscr{I}_d}$.

The generating function $\varphi_I^* = \varphi_{\vec{x},I}^*$, as a function of $(t_J, J \in \mathscr{K}_{\vec{x}_c})$, defined by

$$\varphi_{\vec{x},I}^*(\vec{t}) = \int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p_{\vec{x},I}^*[d\vec{\xi}], \qquad \vec{t} \in \mathbb{C}^{\mathscr{K}_{\vec{x}_c}},$$

is an entire function in $\vec{t}$.

(iii) The set of functions $\varphi_I^* = \varphi_{\vec{x},I}^*$, $I \in \mathscr{K}_{\vec{x}_c}$, are uniquely determined by

$$\begin{aligned} &x_{c,I}\,\frac{\partial \varphi_I^*}{\partial t_J}(\vec{0}) = \Lambda_{IJ}(\vec{x})\, x_J, \qquad \text{if} \quad I, J \in \mathscr{K}_{\vec{x}_c}, \\ &x_{c,I}\varphi_I^*(\lambda\vec{t}) = \Phi_I(\vec{x}_c\, \vec{\varphi}^*(\vec{t})), \quad \vec{t} \in \mathbb{C}^{\mathscr{K}_{\vec{x}_c}}, \qquad \text{if} \quad I \in \mathscr{K}_{\vec{x}_c}, \end{aligned} \tag{25}$$

where we define $\varphi_J^* = 0$ for $J \notin \mathscr{K}_{\vec{x}_c}$, and in the variable for $\vec{\Phi}$ we used an (irregular) notation

$$(\vec{x}\, \vec{\varphi}^*(\vec{t}))_J = x_J\, \varphi_J^*(\vec{t}), \qquad J \in \mathscr{I}_d.$$

(iv) If $\vec{x} \in \mathscr{D}om(\vec{x}_c) \cap \Xi_d$ and $I \in \mathscr{K}_{\vec{x}_c}$, then the distribution of $\lambda^{-n}L(w)$, the scaled length of $w \in W_I^{(n)}$, under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $\bar{p}_{\vec{x},I}^*$, which has a C^∞ density $\bar{\rho}_{\vec{x},I}^*$.

In particular, $\bar{\rho}_{\vec{x},(1)}^*(\xi) > 0$, $\xi > 0$.

We move on to the results on paths with step numbers fixed, instead of paths with endpoints fixed.

We denote the self-avoiding paths starting from origin O by $W^{(0)}$: $W^{(0)} = \{w \in W_0 \mid w(0) = O\}$. Also, we define, in analogy with (15), $W_{\mathscr{K}}^{(0)} = \{w \in W^{(0)} \mid s_J(w) = 0, J \notin \mathscr{K}\}$, for $\mathscr{K} \subset \mathscr{I}_d$.

For each $k \in \mathbb{Z}_+$, let

$$N(k) = \sharp\{w \in W^{(0)} \mid L(w) = k\}$$

be the number of self-avoiding paths of length k starting from O, and for $\mathscr{K} \subset \mathscr{I}_d$,

$$N_{\mathscr{K}}(k) = \sharp\{w \in W_{\mathscr{K}}^{(0)} \mid L(w) = k\}.$$

Theorem 10. If there exists a critical point $\beta_c \in \mathbb{R}$ of the full model, then there exist positive constants C_i, $i = 1, 2$, and real constants C_i, $i = 3, 4$, such that

$$C_1 k^{C_3} e^{\beta_c k} \leqq N(k) \leqq C_2 k^{C_4} e^{\beta_c k}, \qquad k = 1, 2, 3, \ldots.$$

Similarly, if there exists a critical point $\beta_{c,\mathrm{res}}$ of the restricted model, then there exist positive constants C'_i, $i=1,2$, and real constants C'_i, $i=3,4$, such that

$$C'_1 k^{C'_3} e^{\beta_{c,\,\mathrm{res}} k} \leqq N(k) \leqq C'_2 k^{C'_4} e^{\beta_{c,\,\mathrm{res}} k}, \qquad k=1,2,3,\ldots.$$

For each positive integer k, let $\tilde{P}_k$ be a distribution on $W^{(0)}$, defined by

$$\tilde{P}_k[A] = \frac{1}{N(k)} \sharp\{w \in A \mid L(w) = k\}, \qquad A \subset W^{(0)}.$$

The next result shows the existence of the exponent for mean square displacement, which indicates (in a log ratio sense) that a typical self-avoiding path w of length $L(w)=k$ deviates from the starting point by $|w(k)| \asymp k^{1/d_w}$, where

$$d_w = \frac{\log \lambda}{\log 2}. \tag{26}$$

Theorem 11. If there exists a critical point $\beta_c \in \mathbb{R}$ of the full model, then there exist constants α, k_0, C, and C' such that

$$s \log k - s\alpha \log\log k + C \leqq \log E_k[|w(k)|^{s\,d_w}] \leqq s \log k + s\alpha \log\log k + C',$$
$$k \geqq k_0, \quad s \geqq 0,$$

where E_k denotes expectation with respect to $\tilde{P}_k$, and $|\cdot|$ denotes the (Euclidean) length in $\mathbb{R}^d$.

A similar result holds for the restricted model.

Remarks. (i) The intuitive meaning of (26) is as follows. λ is the asymptotic rate of increase of the number of steps as n is increased. Since the size (scale) is increased by a factor 2 as n is increased by 1, the log ratio of the number of steps to the distance scale is equal to the log ratio of λ and 2. Though this is a standard idea in the renormalization group approaches to asymptotic behaviors, our emphasis here is on the precise mathematical statements and rigorous proofs that fit to such intuitive pictures.

(ii) As may be seen from the fact that λ is defined in (FP2) as the largest eigenvalue of the differential map of $\vec{\Phi}$ at $\vec{x}_c$ while β_c is defined in (CS1) as the intersection of the canonical curve and the critical surface, these two quantities have no direct relations. In fact, in the common wisdom of the renormalization group ideas, the exponent for mean square displacement is considered to be universal; i.e., independent of details of the system, (in fact the full model and the restricted model have the same λ), while the connective constant depends on the details of the system (the full model and the restricted model have different values of β_c). This is related to the fact that λ is a solution to an algebraic equation and can be calculated explicitly to arbitrary precision, while β_c has no such simple algebraically closed formula and is difficult to calculate explicitly.

(iii) It may be worthwhile to note that the self-avoiding walks on hypercubic lattice $\mathbb{Z}^n$ in high dimensions ($n > 4$) are proved to be in the same universality class as the random walks—i.e., they have similar asymptotic behaviors—by the lace expansion methods.[8,9] In a sense, the self-avoiding walks in high dimensional spaces may be seen as (non-trivial) perturbations to the random walks.

However, it is also believed (and trivially true for $n = 1$!), that for $n < 4$ the asymptotic behaviors are very different, hence the problem remains in lower dimensional spaces. We note that dSG are, from the renormalization group point of view, spaces "between $\mathbb{Z}^1$ and $\mathbb{Z}^2$." We also point out that the lace expansion method heavily uses translational invariance of $\mathbb{Z}^n$, while fractals lack the invariance. In fact, the random walks and the self-avoiding walks are known to be in different universality classes on 2SG and 3SG; the values of exponents for mean square displacement of the self-avoiding walks and the random walks have no explicit simple relations.[10]

4. OUTLINE OF PROOFS

Given the precise formulation of assumptions and claims in Section 3, we can prove all the Theorems in Section 3.2. However, to reduce the size of this paper, we will only give a brief outline of the proof here, which, with earlier works for $d = 2$[3–5] and $d = 3$,[6] we hope will enable a reader to guess how a proof works. A full detail of the proof is contained in ref. 11.

The archived preprint[11] is based on the original version (submitted to the journal) of the present paper, and is revised in full accordance with the comments from one of the referees who went through the details of the proofs.

4.1. Phase Structure

Theorem 7 is a consequence of the following, which shows the phase structures of the models.

Let

$$D = \{\vec{x} \in \Xi_d \mid \sup_{n \in \mathbb{Z}_+} \max_{I \in \mathscr{I}_d} X_{n,I}(\vec{x}) < \infty\}, \tag{27}$$

and denote its exterior, boundary, interior in Ξ_d by D^c, ∂D, and D^o, respectively. (Namely, $D^c = \Xi_d \setminus \bar{D}$, $\partial D = \bar{D} \cap \overline{D^c}$, $D^o = D \setminus \partial D$.) Let also

$$\tilde{D} = \{\vec{x} \in \Xi_d \mid \lim_{n \to \infty} \max_{I \in \mathscr{I}_d} X_{n,I}(\vec{x}) = 0\}.$$

Theorem 12. (i) It holds that

$$D = \{\vec{x} \in \Xi_d \mid \sup_{n \in \mathbb{Z}_+} \max_{I \in \mathscr{I}_d} X_{n,I}(\vec{x}) \leqq 1\}. \tag{28}$$

In particular, D is a closed subset of Ξ_d.

(ii) Let $\vec{x} \in D$ and $\vec{x}' \in \Xi_d$. If, for each $I \in \mathscr{I}_d$ either $x'_I < x_I$ or $x'_I = x_I = 0$ holds, then $\vec{x}' \in \tilde{D}$.

(iii) It holds that

$$D^o = \tilde{D} \tag{29}$$

(iv) D^c, ∂D, and D^o are non-empty invariant sets of $\vec{\Phi}$.

The proof of this theorem in ref. 11 also shows that

$$\lim_{n \to \infty} X_{n,(1)}(\vec{x}) = \infty, \qquad \vec{x} \in D^c. \tag{30}$$

Proof of Theorem 7. The case $\beta = \beta_c$ holds by the definition (CS1).

By the definitions (CS1) and (FP4), $\lim_{n \to \infty} \vec{X}_n(\vec{x}_{\mathrm{can}}(\beta_c)) = \vec{x}_c \neq \vec{0}$, which, with (27) and (29), implies $\vec{x}_{\mathrm{can}}(\beta_c) \in \partial D$. Monotonicity property in Theorem 12 then implies $\vec{x}_{\mathrm{can}}(\beta) \in \tilde{D}$ if $\beta > \beta_c$, hence, in particular, $\lim_{n \to \infty} \vec{Z}_n(\beta) = 0$.

Finally, if $\beta < \beta_c$ and $\vec{x}_{\mathrm{can}}(\beta) \in D = \bar{D}$, then the monotonicity property in Theorem 12 implies $\vec{x}_{\mathrm{can}}(\beta_c) \in \tilde{D} = D^o$, which contradicts $\vec{x}_{\mathrm{can}}(\beta_c) \in \partial D$. Hence $\vec{x}_{\mathrm{can}}(\beta) \in D^c$ and in particular, with the same argument as that led to (30), we have $\lim_{n \to \infty} \sum_{i=1}^{d} Z_{n,(i)}(\beta) = \infty$.

The case of restricted model is similarly proved, if we note (CS2) in place of (CS1). ▮

4.2. Distribution of Path Length

Theorem 9 is proved along similar lines as in ref. 6, Sect. 4 (but requires additional care, because we here do not use explicit formula for specific d).

That the convergence of generating functions $X_{n,I}$ implies the convergence of distribution of scaled length of paths may be observed by the following simple fact. For $\vec{x} = (x_I, I \in \mathscr{I}_d) \in \mathbb{C}^{\mathscr{I}_d}$ and $\vec{t} = (t_I, I \in \mathscr{I}_d) \in \mathbb{C}^{\mathscr{I}_d}$, we use an (irregular) notation

$$\vec{x}(\vec{t}) = (x_I \exp(\lambda^{-n} t_I), I \in \mathscr{I}_d). \tag{31}$$

Denote by $p_{\vec{x},n,I}$, the joint distribution of $(\lambda^{-n} s_J,\ J \in \mathscr{I}_d)$ under $\mu_{\vec{x},n,I}$. Then its generating function is expressed, with the definitions (9) and (23), as

$$\int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p_{\vec{x},n,I}[d\vec{\xi}] = \frac{X_{n,I}(\vec{x}(\vec{t}))}{X_{n,I}(\vec{x})}, \qquad \vec{t} \in \mathbb{C}^{\mathscr{I}_d}, \tag{32}$$

4.3. Exponent for Mean Square Displacement

As in the case of $d = 3$,[6] to prove Theorem 10 one uses Tauberian type estimates for the number of paths $N(k)$.

Theorem 10 gives sufficient estimate for the denominator $N(k)$ of the expectation $E_k[\,\cdot\,]$ in Theorem 11. To estimate the numerator (hence to prove Theorem 11), one needs two more steps:

- Large deviation type estimates on long paths and short paths.
- Reflection principle.

Among these steps, the reflection principle turns out to be least intuitive in generalizing to arbitrary dimensions. To avoid falling in a pitfall, it is safer to define reflections in an algebraic way, by introducing a natural coordinate system, which we explain in Appendix 4.

5. RESTRICTED MODEL ON THE 4-DIMENSIONAL SIERPIŃSKI GASKET

In this section, we consider the restricted model for $d = 4$. The RG map $\vec{\Phi}$ (Proposition 4) is a map on 6 dimensional space $\mathbb{C}^{\mathscr{I}_4}$ where, as in Section 2, $\mathscr{I}_4 = \{(1), (1,1), (2), (3), (4), (1,2)\}$. (For convenience, we assign the second coordinate to $(1, 1)$ in this section.)

We will consider the restricted self-avoiding walks, the self-avoding paths w starting from O with the property $s_J(w) = 0$, $J \notin \mathscr{K}_{\mathrm{res}}$, where $\mathscr{K}_{\mathrm{res}} = \{(1), (11)\}$ (see (13) and (15)). We regard $\mathbb{R}_+^{\mathscr{K}_{\mathrm{res}}} = \{(x_{(1)}, x_{(11)}, 0, \ldots, 0) \mid x_{(1)}, x_{(11)} \in \mathbb{R}_+\} \subset \mathbb{R}_+^{\mathscr{I}_4}$.

To apply the results of previous sections, we use the following explicit properties of $\vec{\Phi}$.

Proposition 13. The map $\vec{\Phi}$ satisfies the following.

(i) $\vec{\Phi}$ is a 6 dimensional vector valued function whose components are polynomials in 6 variables (1), (11), (2), (3), (4), (12) with positive integer coefficients. The degree of each term in the polynomials are no less than 2 and no greater than 5.

(ii)

$$\begin{aligned}\Phi_{(1)}(x, y, 0, 0, 0, 0) &= x^2+3x^3+6x^4+6x^5+12x^3y+30x^4y+18x^2y^2\\&\quad+78x^3y^2+96x^2y^3+132xy^4+132y^5,\\ \Phi_{(1,1)}(x, y, 0, 0, 0, 0) &= x^4+2x^5+4x^3y+13x^4y+32x^3y^2+88x^2y^3\\&\quad+22y^4+220xy^4+186y^5.\end{aligned} \tag{33}$$

(iii) There exist polynomials $\Phi_{I,i}$, $I=(1),(11)$, $i=0,1,2,3,4$, of positive coefficients, such that $\Phi_{(1),0}$ contains a term $x_{(1)}^2$, $\Phi_{(11),0}$ contains a term $x_{(1)}^4$, and

$$\begin{aligned}\Phi_{(1)}(\vec{x}) &= \Phi_{(1),1}(\vec{x})\,x_{(1)}+\tfrac{1}{2}\,\Phi_{(1),2}(\vec{x})\,x_{(2)}+\tfrac{1}{3}\,\Phi_{(1),3}(\vec{x})\,x_{(3)}\\&\quad+\Phi_{(1),4}(\vec{x})\,x_{(11)}+\Phi_{(1),0}(\vec{x}),\\ \Phi_{(2)}(\vec{x}) &= \Phi_{(1),1}(\vec{x})\,x_{(2)}+\Phi_{(1),2}(\vec{x})\,x_{(3)}+\Phi_{(1),3}(\vec{x})\,x_{(4)}+\Phi_{(1),4}(\vec{x})\,x_{(12)},\\ \Phi_{(11)}(\vec{x}) &= \Phi_{(11),1}(\vec{x})\,x_{(1)}+\tfrac{1}{2}\,\Phi_{(11),2}(\vec{x})\,x_{(2)}+\tfrac{1}{3}\,\Phi_{(11),3}(\vec{x})\,x_{(3)}\\&\quad+\Phi_{(11),4}(\vec{x})\,x_{(11)}+\Phi_{(11),0}(\vec{x}),\\ \Phi_{(12)}(\vec{x}) &= \Phi_{(11),1}(\vec{x})\,x_{(2)}+\Phi_{(11),2}(\vec{x})\,x_{(3)}+\Phi_{(11),3}(\vec{x})\,x_{(4)}+\Phi_{(11),4}(\vec{x})\,x_{(12)}.\end{aligned} \tag{34}$$

(iv) For each $I \notin \mathscr{K}_{\mathrm{res}}$ there exist positive integers $m=m_I$ and $m'=m'_I$ such that $\Phi_{(1)}$ and $\Phi_{(11)}$ contain terms $x_{(1)}^m x_I$ and $x_{(1)}^{m'} x_I$, respectively.

(v) If $I \notin \mathscr{K}_{\mathrm{res}}$, then each term in Φ_I contains a positive power of x_J for some $J \notin \mathscr{K}_{\mathrm{res}}$. Furthermore, each term in $\Phi_{(3)}$ and $\Phi_{(4)}$ has total degree 2 or more of x_J's with $J \notin \mathscr{K}_{\mathrm{res}}$. $\Phi_{(2)}$ contains a term $x_{(1)}^3 x_{(11)} x_{(12)}$ and $\Phi_{(12)}$ contains a term $x_{(1)}^4 x_{(2)}$.

Proposition 13 is proved by explicit calculations on 4SG. The explicit forms (33) are given in ref. 7, Eqs. (A1) and (A2).

Proposition 13 implies, after somewhat lengthy explicit calculations, the following Theorem 14, which assures that the assumptions formulated in Section 3.1 are satisfied for the restricted model on 4SG.

To reduce the length of the present paper, we omit proofs of Proposition 13 and Theorem 14. See ref. 11 for a proof.

Theorem 14. (i) $\Phi_{(1)}(x, y, 0, 0, 0, 0) = x$, and $\Phi_{(11)}(x, y, 0, 0, 0, 0) = y$ has a unique solution $\vec{x}_c = (x_c, y_c, 0, 0, 0, 0)$ in $\{(x, y, 0, 0, 0, 0) \in \mathbb{R}_+^{\mathscr{K}_{\mathrm{res}}} \mid (x, y, 0, 0, 0, 0) \in \Xi_4 \setminus \{\vec{0}\}\}$.

$x_c = 0.326490898\cdots$ and $y_c = 0.027929572\cdots$ are positive. (In particular, $\mathscr{K}_{\mathrm{res}} = \mathscr{K}_{\vec{x}_c}$.)

(ii) $\vec{x}_c$ is a self-avoiding fixed point; i.e., satisfies (FP1)–(FP4).

(iii) There exists a critical point of the restricted model $\beta_{c,\mathrm{res}}$.

Theorem 14 and the results in Section 3.2 imply the following results on the asymptotic behaviors of restricted self-avoiding paths on the 4 dimensional pre-Sierpiński gasket.

Theorem 15. Let $\vec{x}_c = (x_c, y_c, 0, 0, 0, 0) \in \mathbb{R}_+^6$ and $\beta_{c,\mathrm{res}} = \beta_{c,\mathscr{K}_{\mathrm{res}}} \in \mathbb{R}$ be the constants defined in Theorem 14. Then the following holds for the restricted self-avoiding paths on the 4 dimensional pre-Sierpiński gasket.

(i) If $\vec{x} \in \mathscr{D}om(\vec{x}_c)$, then the following hold.

For $I \in \mathscr{K}_{\mathrm{res}} = \{(1), (11)\}$, the joint distribution of scaled generalized path length $(\lambda^{-n}s_{(1)}, \lambda^{-n}s_{(11)})$ under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $p^*_{\vec{x},I}$ on $\mathbb{R}^6$ as $n \to \infty$.

The generating function $\varphi^*_{\vec{x},I}$, defined by

$$\varphi^*_{\vec{x},I}(\vec{t}) = \int_0^\infty e^{\vec{t}\cdot\vec{\xi}} p^*_{\vec{x},I}[d\vec{\xi}], \qquad \vec{t} \in \mathbb{C}^6,$$

is an entire function in $\vec{t}$, and the set of functions $(\varphi^*_{\vec{x},(1)}, \varphi^*_{\vec{x},(11)})$ are uniquely determined by (25) for $d = 4$.

If $\vec{x} \in \mathscr{D}om(\vec{x}_c) \cap \Xi_4$ and $I \in \mathscr{K}_{\mathrm{res}}$, then the distribution of $\lambda^{-n}L(w)$, the scaled length of $w \in W_I^{(n)}$, under $\mu_{\vec{x},n,I}$ converges weakly to a Borel probability measure $\bar{p}^*_{\vec{x},I}$, which has a C^∞ density $\bar{\rho}^*_{\vec{x},I}$.

In particular, $\bar{\rho}^*_{\vec{x},(1)}(\xi) > 0$, $\xi > 0$.

(ii) For $I \in \mathscr{I}_4 = \{(1), (11), (2), (3), (4), (12)\}$,

$$\lim_{n\to\infty} Z_{\mathscr{K}_{\mathrm{res}},n,I}(\beta) = \begin{cases} 0, & \beta > \beta_{c,\mathrm{res}}, \\ x_{c,I}, & \beta = \beta_{c,\mathrm{res}}, \end{cases}$$

and

$$\lim_{n\to\infty} \sum_{i=1}^{4} Z_{\mathscr{K}_{\mathrm{res}},n,(i)}(\beta) = \infty, \qquad \beta < \beta_{c,\mathrm{res}}.$$

(iii) The number $N_{\text{res}}(k) = N_{\mathscr{K}_{\text{res}}}(k)$ of restricted self-avoiding paths of length k starting from 0 satisfies

$$\lim_{k \to \infty} \frac{1}{k} \log N_{\text{res}}(k) = \beta_{c,\,\text{res}}.$$

(iv) The exponent for mean square displacement for the restricted model is the reciprocal of $d_w = \frac{\log \lambda}{\log 2} = 1.6657696\cdots$, in the sense that

$$\lim_{k \to \infty} \frac{1}{\log k} \log E_{\text{res},\,k}[|w(k)|^{s\,d_w}] = s, \qquad s \geqq 0,$$

where $E_{\text{res},\,k}$ is the expectation with respect to the probability measure with equal weight on length k restricted self-avoiding paths starting at O.

Remark. The convergence of $\mu_{\vec{x},\,n,\,I}$ and the properties of the limit measure holds both for the full model and the restricted model (if $\vec{x} \in \mathscr{D}om(\vec{x}_c)$), because they hold independently of (CS2).

APPENDIX A. A COORDINATE SYSTEM FOR *D*-DIMENSIONAL SIERPIŃSKI GASKET AND REFLECTION OPERATOR

Here we introduce a coordinate system with which we define a reflection, which is used in the final step of proving Theorem 11, as outlined in Section 4. See the proof of Theorem 11 in ref. 11 for the actual use (and proofs) of the following.

The definition of pre-dSG in (1) and (2) induces a natural coordinate system on G which is an onto map π: $\{0, 1, 2,..., d\}^{\mathbb{Z}_+} \to G$ defined as follows. For each $v_{0,\,i} = v_i \in G_0$ $(i = 0, 1, 2,..., d)$ we assign a coordinate $(i, 0, 0, 0,...)$;

$$\pi(i, 0, 0, 0,...) = v_{0,\,i}, \qquad i = 0, 1, 2, 3,..., d.$$

For $n = 0, 1, 2,...$ and $i = 0, 1, 2, 3,..., d$, put $G_{n,\,i} = G_n + 2^n v_{0,\,i}$, and define a $1:1$ onto map $\iota_{n,\,i}$: $G_{n,\,i} \to G_n$ by $\iota_{n,\,i}(v) = v - 2^n v_{0,\,i}$. $\iota_{n,\,i}$ naturally induces a $1:1$ onto map $B_{n,\,i} \to B_n$, which we also denote by $\iota_{n,\,i}$.

We proceed with by induction in n and assume that a coordinate system π on G_{n-1} $(= G_{n-1,\,0})$ has been defined for an $n \geqq 1$, in such a way that $\pi(v) \in G_{n-1}$ holds for any $v = (i_0, i_1, i_2,..., i_{n-1}, 0, 0, 0,...)$ with $i_k \in \{0, 1, 2,..., d\}$, $k = 0, 1, 2,..., n-1$. For $v \in G_{n-1,\,j}$, with $j \in \{0, 1, 2,..., d\}$, define

$$\pi(i_0, i_1, i_2,..., i_{n-1}, j, 0, 0,...) = v,$$
$$\text{if} \quad \pi(i_0, i_1, i_2,..., i_{n-1}, 0, 0, 0,...) = \iota_{n-1,\,j}(v).$$

Note that this definition is compatible with $G_{n-1,\,0} = G_{n-1} \subset G_n$.

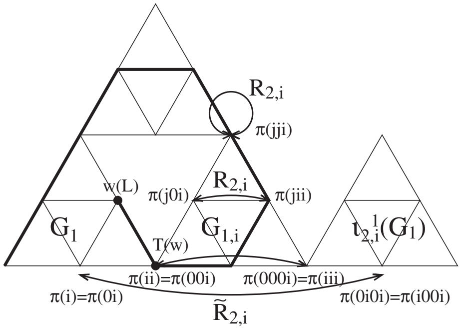

Fig. 1. Reflection $R_{2,i}$ and $\tilde{R}_{2,i}$ in the i–j plane. $\pi(\cdots)$ denotes the point corresponding to the coordinate $(\cdots)$ as defined in the text, and $w(L)$ and $T(w)$ are for the path w indicated by thick lines (see the text for the definitions).

Each point in $G\setminus\{O\}$ has exactly two coordinate representations, because

$$\pi(j, j,..., j, i, 0,...) = \pi(i, i,..., i, j, 0,...) \in G_{m,i} \cap G_{m,j},$$
$$0 \leqq i < j \leqq d, \quad m \in \mathbb{Z}_+.$$

Note also that if $\pi(i_0, i_1, i_2,...) \in G_n$ then $i_k = 0$, $k = n+1, n+2,...$.

We now define a reflection map (see the Fig. 1) with which we formulate a reflection principle in Theorem 17. For each $i = 1, 2, 3,..., d$ define $R_{0,i}$: $\{0, 1, 2,..., d\} \to \{0, 1, 2,..., d\}$ by

$$\begin{cases} R_{0,i}(0) = i, \\ R_{0,i}(i) = 0, \\ R_{0,i}(j) = j, \qquad j \neq 0, i. \end{cases}$$

For $n = 1, 2, 3,...$ and $i = 1, 2,..., d$, we define $1:1$ maps $R_{n,i} : G \to G$ and $\tilde{R}_{n,i}$: $G \to G$ ("partial reflections" with respect to a hyperplane containing $\pi(j, j,..., j, i, 0, 0,...) \in G_{n-1,i}$, $j \neq 0, i$, and "perpendicular to ith axis"), by:

$$R_{n,i}(\pi(x_0, x_1, x_2,...))$$
$$= \begin{cases} \pi(x_0, x_1, x_2,...), & \text{if} \quad \pi(x_0, x_1, x_2,..., x_n,...) \notin G_{n-1,i}, \\ \pi(R_{0,i}(x_0), R_{0,i}(x_1),..., R_{0,i}(x_{n-1}), x_n, 0, 0, 0,...), & \\ \qquad \text{if} \quad \pi(x_0, x_1, x_2,..., x_n,...) \in G_{n-1,i} \quad \text{and} \quad x_n = i, \end{cases} \tag{35}$$

and

$$
\begin{aligned}
&\tilde{R}_{n,i}(\pi(x_0, x_1, x_2, \ldots)) \\
&= \begin{cases} \pi(x_0, x_1, x_2, \ldots), & \text{if} \quad \pi(x_0, x_1, x_2, \ldots, x_n, \ldots) \notin G_{n-1} \cup \iota_{n,i}^{-1}(G_{n-1}), \\ \pi(R_{0,i}(x_0), R_{0,i}(x_1), \ldots, R_{0,i}(x_{n-1}), x_n, R_{0,i}(x_{n+1}), 0, 0, \ldots), \\ \qquad \text{if} \quad \pi(x_0, x_1, x_2, \ldots, x_n, \ldots) \in G_{n-1} \cup \iota_{n,i}^{-1}(G_{n-1}) \quad \text{and} \quad x_n = 0. \end{cases}
\end{aligned} \tag{36}
$$

Note that G_{n-1}, $G_{n-1,i}$, and $\iota_{n,i}^{-1}(G_{n-1})$ are three copies of G_{n-1} aligned in "i th axis" direction, such that

$$
\begin{aligned}
G_{n-1} \cap G_{n-1,i} &= \{\pi(\overset{0,1,2}{0,0,0}, \ldots, 0, \overset{n}{i}, 0, 0, \ldots)\} \\
&= \{\pi(\overset{0,1,2}{i, i, i}, \ldots, \overset{n-1}{i}, 0, 0, 0, \ldots)\},
\end{aligned} \tag{37}
$$

and

$$
\begin{aligned}
G_{n-1,i} \cap \iota_{n,i}^{-1}(G_{n-1}) &= \{\pi(\overset{0,1,2}{0,0,0}, \ldots, 0, \overset{n+1}{i}, 0, 0, \ldots)\} \\
&= \{\pi(\overset{0,1,2}{i, i, i}, \ldots, i, \overset{n}{i}, 0, 0, \ldots)\}.
\end{aligned} \tag{38}
$$

Note also that, by construction all the points in $G_{n-1,i}$ can be written as $\pi(x_0, x_1, \ldots, x_n, 0, 0, \ldots)$ with $x_n = i$, those in G_{n-1} as $\pi(x_0, \ldots, x_n, 0, 0, \ldots)$ with $x_n = 0$, and those in $\iota_{n,i}^{-1}(G_{n-1})$ as $\pi(x_0, \ldots, x_n, x_{n+1}, 0, \ldots)$ with $x_n = 0$ and $x_{n+1} = i$.

Proposition 16. $R_{n,i}$ and $\tilde{R}_{n,i}$ are $1:1$ maps. Moreover, the following hold.

(i) If $x \in G_{n-1}$ then $\tilde{R}_{n,i}(x) \in \iota_{n,i}^{-1}(G_{n-1})$.

(ii) If $(x, y) \in B_{n-1}$, then $(\tilde{R}_{n,i}(x), \tilde{R}_{n,i}(y)) \in B$.

(iii) If $x \in G_{n-1} \cap G_{n-1,i}$ then $R_{n,i}(x) = \tilde{R}_{n,i}(x)$.

(iv) If $x \in G_n$ then $R_{n,i}(x) \in G_n$.

(v) If $x \in G_{n-1,i} \cap \bigcup_{j \neq 0,i} G_{n-1,j}$ then $R_{n,i}(x) = x$.

(vi) If $(x, y) \in B_{n-1,i}$ then $(R_{n,i}(x), R_{n,i}(y)) \in B$.

Define $\nu\colon W^{(0)} \to \mathbb{Z}_+ \cup \{\infty\}$ by

$$
\nu(w) = \min\{n \in \mathbb{Z}_+ \cup \{\infty\} \mid w(k) \in G_n,\ k = 0, 1, 2, \ldots, L(w)\}, \qquad w \in W^{(0)}. \tag{39}
$$

Note the obvious relation

$$2^{\nu(w)-1} < L(w) \leqq d(d+1)^{\nu(w)}, \qquad w \in W^{(0)}. \tag{40}$$

(The second inequality is because there are $(d+1)^n$ unit simplices in F_n, and within each unit simplex a self-avoiding walk can spend at most d steps.)

For each self-avoiding path $w \in W^{(0)}$ satisfying $|w(L(w))| < 2^{\nu(w)-1}$, we want to assign a self-avoiding path $R(w) \in W^{(0)}$ ("reflected path") such that $|R(w)(L(R(w)))| > 2^{\nu(R(w))-1}$ (the left hand side stands for the Euclidean distance of the endpoints of $R(w)$). This is possible using (35) and (36), as follows.

Given $w = (w(0), w(1),..., w(L(w))) \in W^{(0)}$ with the property

$$w(L(w)) \in G_{n-1} \Big\backslash \bigcup_{i=1}^{d} G_{n-1,\,i}, \qquad \text{where} \quad n = \nu(w), \tag{41}$$

we shall define a path $R(w)$ as follows.

Let $T(w)$ be a positive integer satisfying

$$w(k) \in G_{n-1}, \qquad T(w) \leqq k \leqq L(w), \qquad \text{and} \qquad w(T(w)-1) \notin G_{n-1}.$$

The condition (41) implies that such an integer (uniquely) exists. Since $w(T(w)) \in G_{n-1}$, $w(T(w)-1) \in \bigcup_{i=1}^{d} G_{n-1,\,i}$. Let $i(w) \in \{1, 2,..., d\}$ be such that $w(T(w)-1) \in G_{n-1,\,i(w)}$. Clearly, such an integer is also unique. Also the definitions of $T(w)$ and $i(w)$ imply

$$w(T(w)) \in G_{n-1} \cap G_{n-1,\,i(w)}. \tag{42}$$

Define $R(w)$ by

$$R(w)(k) = \begin{cases} R_{n,\,i(w)}(w(k)), & 0 \leqq k < T(w), \\ \tilde{R}_{n,\,i(w)}(w(k)), & T(w) \leqq k \leqq L(w). \end{cases}$$

Theorem 17. For each $w \in W^{(0)}$ satisfying $|w(L(w))| < 2^{\nu(w)-1}$, $R(w) \in W^{(0)}$ (i.e., is a self-avoiding path starting from O) which satisfies $L(R(w)) = L(w)$, $\nu(R(w)) = \nu(w)+1$, and $2^{\nu(w)} < |R(w)(L(w))|$.

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